

Statement of uncertainty principle for quantum measurements in terms of the Rényi entropies

Alexey E. Rastegin

Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20,
Irkutsk 664003, Russia

Abstract

The aim of the work is to give the explicit proofs of the Rényi-entropy uncertainty relations presented in the previous work [A. Rastegin, arXiv:0805.1777]. The relations with both the state-dependent and state-independent entropic bounds are proved. For a pair of POVM measurements the two relations are obtained. The first of them is generalization of the known results, whereas the second is quite alternative. It is shown that both these relations are meaty. The important case of POVM's with one-rank elements is extra discussed. The measurements designed for distinction between two non-orthogonal quantum states are considered as examples.

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1 Introduction

The Heisenberg uncertainty principle [1] is primary and most known of those results that emboss the fundamental distinctions of the quantum world from the classical world. Conceptual development of quantum theory has lead to a number of related conclusions such as the Bell inequalities [2, 3], the quantum Zeno effect [4], the no-cloning theorem [5], the interaction-free measurement [6, 7], the no-deleting principle [8] and the no-hiding theorem [9]. The general quantitative form of the uncertainty principle was given by Robertson [10].

Suppose A and B are two observables measured in the quantum state ψ . The standard deviations ΔA and ΔB of the two probability distributions then satisfy [10]

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \psi, [A, B] \psi \rangle|, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. This quantitative formulation of the uncertainty principle is best known among physicists. Due to a variety of measurement scenarios, many relations have been stated in more detailed terms [11, 12, 13]. For example, Bohr's principle of complementarity [14] has been quantified by uncertainty relations (see papers [15, 16, 17] and references therein).

The entropic uncertainty relations provide an alternative way to express quantitatively the Heisenberg principle. In many instances these relations are more useful characterization. The first entropic relation was proposed by Hirschman [18]. Namely, he obtained the relation for position and momentum in terms of the Shannon entropies. Hirschman also conjectured an improvement of his result. This conjecture has been proved by Beckner [19] and by Bialynicki-Birula and Mycielski [20]. The concrete calculations of the position and momentum Shannon entropies for the harmonic oscillator and the hydrogen atom have been made by the writers of Ref. [21]. But important as the case of position-momentum is, it is not able to give understanding the limitations on the information characteristics of measurements in all respects.

In the general context, entropic formulation of the uncertainty principle was considered by Deutsch [22]. He emphasized that the right-hand side of Eq. (1.1) is not a fixed lower bound but is itself a function of ψ . Indeed, if the state ψ is an eigenstate of A then $\Delta A = 0$ and the right-hand side of Eq. (1.1) vanishes. So no bound on ΔB is imposed by Eq. (1.1) [23]. Deutsch obtained a state-independent lower bound on the sum of the Shannon entropies of two probability distributions generated by measurement of two observables without degeneracy. Of late years, many other investigations of entropic uncertainty relations have been made. The list of corresponding references can be found in the previous work of the present author [24]. Here we just mention only several papers of interest.

It turned out that the entropic uncertainty relation given in Ref. [22] can significantly be improved. The sharpened relation has been conjectured by Kraus [25] and then established by Maassen and Uffink [23]. However, the formulation stated in Ref. [23] deals with two non-degenerate observables.

A relevant extension to the case of two degenerate observables has been obtained by Krishna and Parthasarathy [26]. Using Naimark's theorem, they also generalized their entropic uncertainty relation to the case of a pair of arbitrary measurements. The entropic uncertainty relations for sets of $N + 1$ complementary observables in N -dimensional Hilbert space were obtained [27, 28]. In more recent paper [29], Massar briefly considered the entropic uncertainty relation in terms of the Shannon entropies for POVM's, whose elements are all rank one [29].

Together with the Shannon entropy, other information entropies are extensively used in the literature. One of them is the Rényi entropy. In Ref. [30], Larsen derived uncertainty relations in terms of the so-called purities, which are directly connected with Rényi's entropy of order two. Białynicki-Birula obtained the uncertainty relations in terms of Rényi's entropies for the position-momentum and angle-angular momentum pairs [31]. In the previous work [24], the present author have posed uncertainty relations for a pair of arbitrary measurements and for a single measurement in the form of inequalities using the Rényi entropies. The aim of the given work is to ensure careful proofs of these relations. In addition, more information on the subject of the entropic uncertainty relations is provided.

2 Background

We shall now describe the notation that is used throughout the text. By \mathcal{H} we denote finite-dimensional Hilbert space. A state of quantum system is described by density matrix. Recall that a density matrix is positive semidefinite matrix with unit trace. Let $\alpha > 0$ and $\alpha \neq 1$; then the Rényi entropy of order α of probability distribution $\{p_i\}$ is defined by [32]

$$H_\alpha(p) := \frac{1}{1-\alpha} \ln \left\{ \sum_i p_i^\alpha \right\} . \quad (2.1)$$

This information measure is a nonincreasing function of order α ; that is, if $\alpha < \beta$ then $H_\alpha \geq H_\beta$ [32]. The limit $\alpha \rightarrow 1$ recovers the Shannon entropy

$$H_1(p) := - \sum_i p_i \ln p_i . \quad (2.2)$$

In the following, orders of Rényi's entropies are assumed to be different from one. The relations for the Shannon entropies can thereupon be obtained by

taking the limit $\alpha \rightarrow 1$ in the final inequalities. Unlike the Shannon entropy, the Rényi entropy is not a concave function of the probability distribution. More precisely, for $\alpha > 1$ the Rényi entropy $H_\alpha(p)$ is not purely convex nor purely concave [33].

A generalized quantum measurement is described by "Positive Operator-Valued Measure" (POVM). This is a set $\{\mathbf{M}_i\}$ of positive semidefinite matrices satisfying [34, 35]

$$\sum_i \mathbf{M}_i = \mathbf{I} , \quad (2.3)$$

where \mathbf{I} is the identity matrix. For given measurement $\{\mathbf{M}_i\}$ and quantum state ρ , the probability of i th outcome is equal to [34, 35]

$$p_i = \text{tr}\{\mathbf{M}_i \rho\} . \quad (2.4)$$

In the mathematical literature, such a set $\{\mathbf{M}_i\}$ is often called "generalized resolution of the identity" for the space \mathcal{H} (for a discussion, see Refs. [36, 37]). In the particular case of orthogonal projections, one is called "orthogonal resolution of the identity" [36, 37].

The Rényi entropy $H_\alpha(\mathbf{M}|\rho)$ of generated probability distribution is then defined by Eqs. (2.1) and (2.4). When a quantum state is pure, that is $\rho = \boldsymbol{\psi} \boldsymbol{\psi}^\dagger$ and $\|\boldsymbol{\psi}\| = 1$, we will write $H_\alpha(\mathbf{M}|\boldsymbol{\psi})$. In this case,

$$\text{tr}\{\mathbf{M}_i \rho\} = \langle \boldsymbol{\psi} , \mathbf{M}_i \boldsymbol{\psi} \rangle . \quad (2.5)$$

Let $\{\mathbf{M}_i\}$ and $\{\mathbf{N}_j\}$ be two POVM's, and $\boldsymbol{\psi}$ a pure state. By definition, we put the function

$$f(\mathbf{M}, \mathbf{N}|\boldsymbol{\psi}) := \max_{ij} \|\mathbf{M}_i^{1/2} \boldsymbol{\psi}\|^{-1} \|\mathbf{N}_j^{1/2} \boldsymbol{\psi}\|^{-1} |\langle \mathbf{M}_i \boldsymbol{\psi} , \mathbf{N}_j \boldsymbol{\psi} \rangle| , \quad (2.6)$$

where the maximum is taken over those values of labels i and j that satisfy $\|\mathbf{M}_i^{1/2} \boldsymbol{\psi}\| \neq 0$ and $\|\mathbf{N}_j^{1/2} \boldsymbol{\psi}\| \neq 0$. In the case of mixed state ρ with the spectral decomposition

$$\rho = \sum_\lambda \lambda \boldsymbol{\psi}_\lambda \boldsymbol{\psi}_\lambda^\dagger \quad (2.7)$$

we further define

$$f(\mathbf{M}, \mathbf{N}|\rho) := \max_\lambda f(\mathbf{M}, \mathbf{N}|\boldsymbol{\psi}_\lambda) . \quad (2.8)$$

In order to prove the entropic relation, we shall use Riesz's theorem. A version of Riesz's theorem is posed as follows [38]. (The below formulation is slightly modified in comparison with the one given in Ref. [38].) Let $\mathbf{x} \in \mathbb{C}^n$

be n -tuple of complex numbers x_j and let t_{ij} be entries of matrix \mathbf{T} of order $m \times n$. Define η to be maximum of the set $|t_{ij}|$, that is

$$\eta := \max_{ij} |t_{ij}| . \quad (2.9)$$

To each \mathbf{x} assign m -tuple $\mathbf{y} \in \mathbb{C}^m$ with elements

$$y_i(x) := \sum_{j=1}^n t_{ij} x_j \quad (i = 1, \dots, m) . \quad (2.10)$$

So the fixed matrix \mathbf{T} describes a linear transformation $\mathbb{C}^n \rightarrow \mathbb{C}^m$. For any $b \geq 1$ we also define

$$S_b(x) := \left\{ \sum_j |x_j|^b \right\}^{1/b} . \quad (2.11)$$

Lemma 1 *Suppose the matrix \mathbf{T} satisfies*

$$\sum_i |y_i|^2 \leq \sum_j |x_j|^2 \quad (2.12)$$

for all $\mathbf{x} \in \mathbb{C}^n$; then

$$S_a(y) \leq \eta^{(2-b)/b} S_b(x) , \quad (2.13)$$

where $1/a + 1/b = 1$ and $1 < b < 2$.

In Ref. [38] this result is appeared as theorem 297. Note that Riesz's theorem has been extended to infinite-dimensional spaces by Thorin. In functional analysis one is known as the Riesz-Thorin interpolation theorem [39]. Of course, the above statement can be obtained from the Riesz-Thorin theorem. The needed reasons are contained in Ref. [26] (see the proof of theorem 2.1 therein). But the authors of Ref. [26] do not formulate the above statement explicitly as an individual result.

3 Projective measurements

A projective measurement is described by "Projector-Valued Measure" (PVM). This is a set $\{\mathbf{P}_i\}$ of Hermitian matrices satisfying the property

$$\mathbf{P}_i \mathbf{P}_k = \delta_{ik} \mathbf{P}_i , \quad (3.1)$$

where δ_{ik} is the Kronecker delta, and the completeness relation

$$\sum_i \mathbf{P}_i = \mathbf{I} . \quad (3.2)$$

The two PVM's $\{P_i\}$ and $\{Q_j\}$ generate two probability distributions. Due to the properties of projectors, the probabilities are rewritten as

$$p_i^{(\psi)} = \langle P_i \psi, P_i \psi \rangle , \quad (3.3)$$

$$q_j^{(\psi)} = \langle Q_j \psi, Q_j \psi \rangle , \quad (3.4)$$

The proof of the following statement is given in Appendix A.

Proposition 2 *For two projective measurements $\{P_i\}$ and $\{Q_j\}$ and pure state $\psi \in \mathcal{H}$,*

$$H_\alpha(P|\psi) + H_\beta(Q|\psi) \geq -2 \ln f(P, Q|\psi) , \quad (3.5)$$

where orders α and β satisfy $1/\alpha + 1/\beta = 2$.

As it is mentioned above, the Rényi entropy is not generally concave. Therefore, the lower bound (3.5) cannot directly be extended to the case of mixed state. The Minkowski inequality, which is very helpful result [38], allows to reach the aim. To clarify the exposition, we consider the mixed state σ with the spectral decomposition

$$\sigma = \lambda \psi \psi^\dagger + (1 - \lambda) \varphi \varphi^\dagger , \quad (3.6)$$

where $0 < \lambda < 1$. For the given state σ , the corresponding probabilities are then rewritten as

$$p_i = \text{tr}\{P_i \sigma\} = \lambda p_i^{(\psi)} + (1 - \lambda) p_i^{(\varphi)} , \quad (3.7)$$

$$q_j = \text{tr}\{Q_j \sigma\} = \lambda q_j^{(\psi)} + (1 - \lambda) q_j^{(\varphi)} . \quad (3.8)$$

Here the values $p_i^{(\varphi)}$ and $q_j^{(\varphi)}$ are defined by substituting φ for ψ into Eqs. (3.3) and (3.4) respectively. The argumentation of Appendix A, including Eqs. (A.11) and (A.12), are valid for both the pure states ψ and φ . So, under the same conditions on α and β , we can write down

$$\lambda S_\alpha\{p^{(\psi)}\} \leq \eta^{2(1-\beta)/\beta} \lambda S_\beta\{q^{(\psi)}\} , \quad (3.9)$$

$$(1 - \lambda) S_\alpha\{p^{(\varphi)}\} \leq \eta^{2(1-\beta)/\beta} (1 - \lambda) S_\beta\{q^{(\varphi)}\} , \quad (3.10)$$

where η is now equal to $f(P, Q|\sigma)$, that is the maximum among $f(P, Q|\psi)$ and $f(P, Q|\varphi)$. On this stage the Minkowski inequality should be used. By $\alpha > 1$ and $\beta < 1$, there hold

$$S_\alpha\{\lambda p^{(\psi)} + (1 - \lambda) p^{(\varphi)}\} \leq \lambda S_\alpha\{p^{(\psi)}\} + (1 - \lambda) S_\alpha\{p^{(\varphi)}\} , \quad (3.11)$$

$$\lambda S_\beta\{q^{(\psi)}\} + (1 - \lambda) S_\beta\{q^{(\varphi)}\} \leq S_\beta\{\lambda q^{(\psi)} + (1 - \lambda) q^{(\varphi)}\} . \quad (3.12)$$

Summing Eqs. (3.9) and (3.10), due to (3.11) and (3.12) we finally get the same relation (A.11) in which the probabilities are already defined by Eqs. (3.7) and (3.8). By those transformations that have lead to Eq. (A.15), we obtain an entropic relation

$$H_\alpha(\mathbf{P}|\sigma) + H_\beta(\mathbf{Q}|\sigma) \geq -2 \ln f(\mathbf{P}, \mathbf{Q}|\sigma) . \quad (3.13)$$

The case of mixed state ρ with the spectral decomposition (2.7) can be considered in the same manner. Then the following statement takes place.

Proposition 3 *For two projective measurements $\{\mathbf{P}_i\}$ and $\{\mathbf{Q}_j\}$ and any mixed state ρ ,*

$$H_\alpha(\mathbf{P}|\rho) + H_\beta(\mathbf{Q}|\rho) \geq -2 \ln f(\mathbf{P}, \mathbf{Q}|\rho) , \quad (3.14)$$

where orders α and β satisfy $1/\alpha + 1/\beta = 2$.

4 One of measurement is generalized

In this section the above result will be extended to the case when one of two measurement is described by POVM. Elaborating the ideas of Ref. [26], we shall use the Naimark extension. All the necessary details are gathered in Appendix B. The following statement takes place.

Proposition 4 *Let $\{\mathbf{M}_i\}$ be a POVM measurement, and let $\{\mathbf{Q}_j\}$ be a PVM measurement. Then for any mixed state ρ*

$$H_\alpha(\mathbf{M}|\rho) + H_\beta(\mathbf{Q}|\rho) \geq -2 \ln f(\mathbf{M}, \mathbf{Q}|\rho) , \quad (4.1)$$

where orders α and β satisfy $1/\alpha + 1/\beta = 2$.

Proof Substituting \mathbf{M}_i for \mathbf{E}_i and \mathbf{Q}_j for \mathbf{G}_j in the formulas of Appendix B, we will consider the two measurements $\{\tilde{\mathbf{M}}_i\}$ and $\{\tilde{\mathbf{Q}}_j\}$ in the enlarged space $\tilde{\mathcal{H}}$. The measurement $\{\tilde{\mathbf{M}}_i\}$ is projective due to the Naimark theorem. The measurement $\{\tilde{\mathbf{Q}}_j\}$ is projective, because the measurement $\{\mathbf{Q}_j\}$ is projective. By the statement of Proposition 3, we then have

$$H_\alpha(\tilde{\mathbf{M}}|\tilde{\omega}) + H_\beta(\tilde{\mathbf{Q}}|\tilde{\omega}) \geq -2 \ln f(\tilde{\mathbf{M}}, \tilde{\mathbf{Q}}|\tilde{\omega}) \quad (4.2)$$

for arbitrary mixed state $\tilde{\omega}$ in the enlarged space $\tilde{\mathcal{H}}$. To each density matrix of the form (2.7) assign the density matrix

$$\tilde{\rho} = \sum_\lambda \lambda \tilde{\psi}_\lambda \tilde{\psi}_\lambda^\dagger , \quad (4.3)$$

where state vector $\tilde{\boldsymbol{\psi}}_\lambda$ is defined by

$$\tilde{\boldsymbol{\psi}}_\lambda := \begin{bmatrix} \boldsymbol{\psi}_\lambda \\ \mathbf{0} \end{bmatrix} . \quad (4.4)$$

In the particular case of state $\tilde{\rho}$ the relation (4.2) is clearly valid. The $\boldsymbol{\psi}_\lambda$'s form the orthonormal set in the space \mathcal{H} . Hence we obtain

$$\tilde{\boldsymbol{\psi}}_\lambda^\dagger \tilde{\boldsymbol{\psi}}_\mu = [\boldsymbol{\psi}_\lambda^\dagger \quad \mathbf{0}] \begin{bmatrix} \boldsymbol{\psi}_\mu \\ \mathbf{0} \end{bmatrix} = \boldsymbol{\psi}_\lambda^\dagger \boldsymbol{\psi}_\mu = \delta_{\lambda\mu} . \quad (4.5)$$

So the $\tilde{\boldsymbol{\psi}}_\lambda$'s form the (incomplete) orthonormal set in the space $\tilde{\mathcal{H}}$. Due to this fact and the properties of the trace,

$$\text{tr}\{\tilde{\mathbf{M}}_i \tilde{\rho}\} = \sum_\lambda \lambda \langle \tilde{\boldsymbol{\psi}}_\lambda, \tilde{\mathbf{M}}_i \tilde{\boldsymbol{\psi}}_\lambda \rangle = \sum_\lambda \lambda \langle \boldsymbol{\psi}_\lambda, \mathbf{M}_i \boldsymbol{\psi}_\lambda \rangle = \text{tr}\{\mathbf{M}_i \rho\} , \quad (4.6)$$

where we use Eq. (B.8). By a similar argument with Eq. (B.12),

$$\text{tr}\{\tilde{\mathbf{Q}}_j \tilde{\rho}\} = \text{tr}\{\mathbf{Q}_j \rho\} . \quad (4.7)$$

In other words, we have $\tilde{p}_i = p_i$ and $\tilde{q}_j = q_j$ for any state of the form (4.3). Therefore, the corresponding Rényi entropies are related by

$$H_\alpha(\tilde{\mathbf{M}}|\tilde{\rho}) = H_\alpha(\mathbf{M}|\rho) , \quad (4.8)$$

$$H_\beta(\tilde{\mathbf{Q}}|\tilde{\rho}) = H_\beta(\mathbf{Q}|\rho) . \quad (4.9)$$

By the relevant substitutions into Eq. (B.18) and the definition (2.8),

$$f(\tilde{\mathbf{M}}, \tilde{\mathbf{Q}}|\tilde{\rho}) = f(\mathbf{M}, \mathbf{Q}|\rho) . \quad (4.10)$$

The last three equalities are valid for arbitrary matrix of the form (4.3) and, therefore, for arbitrary matrix of the form (2.7). But the latter is general form of density matrix on the space \mathcal{H} . So from Eq. (4.2) we immediately obtain Eq. (4.1). ■

5 The main results

In this section the entropic uncertainty relations for general case will be established. We shall first obtain a lower bound on the sum of entropies of

two POVM measurements $\{\mathbf{M}_i\}$ and $\{\mathbf{N}_j\}$. Following the argumentation of the previous section, let us substitute \mathbf{M}_i for \mathbf{E}_i and \mathbf{N}_j for \mathbf{G}_j in the formulas of Appendix B. So we will consider the two measurements $\{\tilde{\mathbf{M}}_i\}$ and $\{\tilde{\mathbf{N}}_j\}$ in the enlarged space $\tilde{\mathcal{H}}$. The measurement $\{\tilde{\mathbf{M}}_i\}$ is now projective due to the Naimark theorem. In general, the measurement $\{\tilde{\mathbf{N}}_j\}$ is not projective. By the statement of Proposition 4, for PVM $\{\tilde{\mathbf{M}}_i\}$ and POVM $\{\tilde{\mathbf{N}}_j\}$ we have

$$H_\alpha(\tilde{\mathbf{M}}|\tilde{\omega}) + H_\beta(\tilde{\mathbf{N}}|\tilde{\omega}) \geq -2 \ln f(\tilde{\mathbf{M}}, \tilde{\mathbf{N}}|\tilde{\omega}) . \quad (5.1)$$

Here $1/\alpha + 1/\beta = 2$ and $\tilde{\omega}$ denotes an arbitrary mixed state in the enlarged space $\tilde{\mathcal{H}}$. Replacing $\tilde{\mathbf{Q}}_j$ with $\tilde{\mathbf{N}}_j$ in Eqs. (4.7), (4.9) and (4.10), we get corresponding equalities for the considered case. That is,

$$H_\beta(\tilde{\mathbf{N}}|\tilde{\rho}) = H_\beta(\mathbf{N}|\rho) , \quad (5.2)$$

$$f(\tilde{\mathbf{M}}, \tilde{\mathbf{N}}|\tilde{\rho}) = f(\mathbf{M}, \mathbf{N}|\rho) \quad (5.3)$$

for arbitrary matrix of the form (4.3). Due to Eqs. (4.8), (5.2) and (5.3), from Eq. (5.1) we immediately obtain the following result.

Theorem 5 *Let $\{\mathbf{M}_i\}$ and $\{\mathbf{N}_j\}$ be two POVM measurements. Then for arbitrary mixed state ρ , there holds*

$$H_\alpha(\mathbf{M}|\rho) + H_\beta(\mathbf{N}|\rho) \geq -2 \ln f(\mathbf{M}, \mathbf{N}|\rho) , \quad (5.4)$$

where orders α and β satisfy $1/\alpha + 1/\beta = 2$.

The statement of Theorem 5 is generalization of Theorem 2.5 of Ref. [26] in the following two respects. First, this result deals with the Rényi entropies instead of the Shannon entropies. Second, it is established for arbitrary mixed state. We shall now obtain the entropic uncertainty relation for a single POVM presented in Ref. [24]. Suppose that $\alpha > 1$. For arbitrary state ρ we then have

$$\sum_j p_i^\alpha \leq p_{\max}^{\alpha-1} \sum_j p_i = p_{\max}^{\alpha-1} ,$$

where p_{\max} is the largest among the probabilities p_i and the normalization condition is used. Hence due to Eq. (2.1) we obtain

$$H_\alpha(\mathbf{M}|\rho) \geq -\ln \phi(\mathbf{M}|\rho) , \quad (5.5)$$

where by definition

$$\phi(\mathbf{M}|\rho) := \max_i \text{tr}\{\mathbf{M}_i \rho\} \equiv p_{\max} . \quad (5.6)$$

Because the Rényi entropy is a nonincreasing function of order α , Equation (5.5) remains valid for $\alpha < 1$. So we at once get the needed relation.

Theorem 6 *Let $\{\mathbf{M}_i\}$ be a POVM measurement. Then for arbitrary mixed state ρ and any order $\alpha > 0$,*

$$H_\alpha(\mathbf{M}|\rho) \geq -\ln \phi(\mathbf{M}|\rho) . \quad (5.7)$$

The both lower bounds in Eqs. (5.4) and (5.7) are dependent on the state in which a quantum system was before measurement. It is easy to obtain state-independent bounds. Such a form of entropic bound is usually discussed in the literature. Let us define the norm of operator \mathbf{A} by

$$\|\mathbf{A}\| := \max_{\|\psi\|=1} \|\mathbf{A}\psi\| . \quad (5.8)$$

As it is shown in Ref. [26], there holds

$$|\langle \mathbf{M}_i \psi, \mathbf{N}_j \psi \rangle| \leq \|\mathbf{M}_i^{1/2} \mathbf{N}_j^{1/2}\| \|\mathbf{M}_i^{1/2} \psi\| \|\mathbf{N}_j^{1/2} \psi\| . \quad (5.9)$$

By definition, we put

$$\bar{f}(\mathbf{M}, \mathbf{N}) := \max_{ij} \|\mathbf{M}_i^{1/2} \mathbf{N}_j^{1/2}\| . \quad (5.10)$$

It follows from Eqs. (2.6), (2.8) and (5.9), that

$$f(\mathbf{M}, \mathbf{N}|\rho) \leq \bar{f}(\mathbf{M}, \mathbf{N}) . \quad (5.11)$$

Using Eqs. (5.4) and (5.11), we then obtain the desired bound.

Corollary 7 *Let $\{\mathbf{M}_i\}$ and $\{\mathbf{N}_j\}$ be two POVM measurements. For arbitrary mixed state ρ there holds*

$$H_\alpha(\mathbf{M}|\rho) + H_\beta(\mathbf{N}|\rho) \geq -2 \ln \bar{f}(\mathbf{M}, \mathbf{N}) , \quad (5.12)$$

where orders α and β satisfy $1/\alpha + 1/\beta = 2$.

In the particular case of Shannon entropies this relation was stated in Ref. [26], for one-rank projectors it reduces to the result given by Maassen and Uffink [23]. The entropic relations (5.4) and (5.12) have been proved under the condition $1/\alpha + 1/\beta = 2$. Suppose now that orders α and β are arbitrary. Due to Eq. (5.7) we can still pose the following uncertainty relation.

Corollary 8 *Let $\{\mathbf{M}_i\}$ and $\{\mathbf{N}_j\}$ be two POVM's. For any mixed state ρ and arbitrary orders $\alpha, \beta \in (0; +\infty)$,*

$$H_\alpha(\mathbf{M}|\rho) + H_\beta(\mathbf{N}|\rho) \geq -\ln [\phi(\mathbf{M}|\rho)\phi(\mathbf{N}|\rho)] . \quad (5.13)$$

Note that there is a natural generalization of Eq. (5.13) to more than two POVM's. Finally, we will obtain a state-independent bound for a single POVM. Due to the Cauchy-Schwarz inequality and the definition (5.8),

$$|\langle \psi, \mathbf{M}_i \psi \rangle| \leq \|\psi\| \|\mathbf{M}_i \psi\| \leq \|\mathbf{M}_i\| \quad (5.14)$$

for any normalized state ψ . Let us put the function

$$\bar{\phi}(\mathbf{M}) := \max_i \|\mathbf{M}_i\| . \quad (5.15)$$

By spectral decomposition of ρ , the linearity of the trace and Eq. (5.14),

$$\phi(\mathbf{M}|\rho) \leq \bar{\phi}(\mathbf{M}) . \quad (5.16)$$

Using Eqs. (5.7) and (5.16), we lastly obtain

$$H_\alpha(\mathbf{M}|\rho) \geq -\ln \bar{\phi}(\mathbf{M}) . \quad (5.17)$$

All the state-dependent and state-independent bounds on the Rényi entropies proved in this section have first been claimed by the present author without proofs [24]. So the above material is supplementary. In Ref. [24], the entropic bounds are illustrated on the example of distinction between non-orthogonal quantum states. In the following, we shall continue examination of the obtained entropic relations.

6 Discussion

We shall now consider some features of the entropic uncertainty relations obtained above. As it is pointed out in Refs. [22, 23], the dependence of the bound in Eq. (1.1) on state ψ leads to some shortcoming. In a certain sense, the state-dependent entropic bounds in Eqs. (5.4) and (5.7) are free from this defect. That is, if $\bar{f}(\mathbf{M}, \mathbf{N}) < 1$ then for each state ρ the bound (5.4) is nonzero due to Eq. (5.11). Further, if $\bar{\phi}(\mathbf{M}) < 1$ then for any ρ the bound (5.7) is nonzero due to Eq. (5.16). This situation takes place if and

only if each of POVM elements has only those eigenvalues that are strictly less than 1. In physical applications such a property usually implies that no POVM elements are projectors. This is sufficiently common case. In fact, the norm of each projector is equal to 1. Further, due to Eq. (2.3) the operator $(\mathbf{I} - \mathbf{M}_k)$ is positive semidefinite for any fixed k . Hence we have $\|\mathbf{M}_k\| \leq 1$, that is no eigenvalues of \mathbf{M}_k exceed 1. In addition, both the state-dependent bounds (5.4) and (5.7) can be stronger than the state-independent bounds (5.12) and (5.13) respectively.

It must be stressed that inequality (5.11) is always saturated for two POVM's consisting of elements of rank one only. Due to the Davies theorem [42], such measurements are sufficient to maximize the mutual information. We shall now prove that for POVM's with only one-dimensional operators the equality holds in Eq. (5.11) regardless of state ρ . Let us assume that

$$\mathbf{M}_i = \mu_i \mathbf{m}_i \mathbf{m}_i^\dagger, \quad (6.1)$$

$$\mathbf{N}_j = \nu_j \mathbf{n}_j \mathbf{n}_j^\dagger, \quad (6.2)$$

for all values of labels i and j . (Note that no summation is taken in Eqs. (6.1) and (6.2).) Here $0 \leq \mu_i \leq 1$, $0 \leq \nu_j \leq 1$, and the vectors \mathbf{m}_i and \mathbf{n}_j are all normalized. For arbitrary pure state $\boldsymbol{\psi}$, we then have

$$\|\mathbf{M}_i^{1/2} \boldsymbol{\psi}\| = \mu_i^{1/2} |\langle \mathbf{m}_i, \boldsymbol{\psi} \rangle|, \quad (6.3)$$

$$\|\mathbf{N}_j^{1/2} \boldsymbol{\psi}\| = \nu_j^{1/2} |\langle \mathbf{n}_j, \boldsymbol{\psi} \rangle|. \quad (6.4)$$

Next, we get $|\langle \mathbf{M}_i \boldsymbol{\psi}, \mathbf{N}_j \boldsymbol{\psi} \rangle| = \mu_i \nu_j |\langle \boldsymbol{\psi}, \mathbf{m}_i \rangle \langle \mathbf{m}_i, \mathbf{n}_j \rangle \langle \mathbf{n}_j, \boldsymbol{\psi} \rangle|$. Therefore, for any $\boldsymbol{\psi}$ there holds

$$\|\mathbf{M}_i^{1/2} \boldsymbol{\psi}\|^{-1} \|\mathbf{N}_j^{1/2} \boldsymbol{\psi}\|^{-1} |\langle \mathbf{M}_i \boldsymbol{\psi}, \mathbf{N}_j \boldsymbol{\psi} \rangle| = \sqrt{\mu_i \nu_j} |\langle \mathbf{m}_i, \mathbf{n}_j \rangle|. \quad (6.5)$$

The latter is simply equal to $\|\mathbf{M}_i^{1/2} \mathbf{N}_j^{1/2}\|$ accordingly the properties of operator norm and Eqs. (6.1) and (6.2). Combining this with the definitions (2.8) and (5.10) finally gives the claimed equality.

Following Ref. [24], we consider the example of discriminating between pure states $\boldsymbol{\psi}_1 \equiv \mathbf{e}_0$ and $\boldsymbol{\psi}_2 \equiv (\mathbf{e}_0 + \mathbf{e}_1)/\sqrt{2}$, where \mathbf{e}_0 and \mathbf{e}_1 are two orthonormal vectors. This example is very particular case of the quantum hypothesis testing [43]. In the Helstrom scheme [34, 36], which is not error-free, the optimal measurement is described by PVM $\{\mathbf{N}_1, \mathbf{N}_2\}$ with elements $\mathbf{N}_1 = \mathbf{u} \mathbf{u}^\dagger$ and $\mathbf{N}_2 = \mathbf{v} \mathbf{v}^\dagger$, where

$$\mathbf{u} \equiv \cos(\pi/8) \mathbf{e}_0 - \sin(\pi/8) \mathbf{e}_1, \quad (6.6)$$

$$\mathbf{v} \equiv \sin(\pi/8) \mathbf{e}_0 + \cos(\pi/8) \mathbf{e}_1. \quad (6.7)$$

In the error-free discrimination scheme [44, 45, 46] the optimal measurement is described by POVM $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ with elements

$$\mathbf{M}_1 = 2^{-1/2}(\sqrt{2} + 1)^{-1} (\mathbf{e}_0 - \mathbf{e}_1) (\mathbf{e}_0 - \mathbf{e}_1)^\dagger, \quad (6.8)$$

$$\mathbf{M}_2 = \sqrt{2} (\sqrt{2} + 1)^{-1} \mathbf{e}_1 \mathbf{e}_1^\dagger, \quad (6.9)$$

$$\mathbf{M}_3 = \mathbf{1} - \mathbf{M}_1 - \mathbf{M}_2. \quad (6.10)$$

The elements of both the above POVM's are all rank one. Due to this fact and $\bar{f}(\mathbf{M}, \mathbf{N})^2 = 1/2$ [24], Theorem 5 gives

$$H_\alpha(\mathbf{M}|\rho) + H_\beta(\mathbf{N}|\rho) \geq \ln 2 \quad (6.11)$$

for any state ρ . Further, $\phi(\mathbf{M}|\psi_1) = 2^{-1/2}$ and $\phi(\mathbf{N}|\psi_1) = 2^{-3/2}(\sqrt{2} + 1)$ by calculations [24]. Corollary 8 then gives

$$H_\alpha(\mathbf{M}|\psi_1) + H_\beta(\mathbf{N}|\psi_1) \geq \ln 4 - \ln(\sqrt{2} + 1). \quad (6.12)$$

The right-hand side of Eq. (6.11) is greater than the right-hand side of Eq. (6.12). Thus, for state ψ_1 the entropic relation (5.4) provides more stronger bound than the entropic relation (5.13). The difference between these bounds is $\ln(\sqrt{2} + 1) - \ln 2 \approx 0.188$. In turn, the entropic relation (5.13) can be stronger than the entropic relation (5.4). Let us consider the eigenstate φ_3 of operator \mathbf{M}_3 which is expressed as

$$\varphi_3 = 2^{-3/4} \left\{ (\sqrt{2} + 1)^{1/2} \mathbf{e}_0 + (\sqrt{2} - 1)^{1/2} \mathbf{e}_1 \right\}. \quad (6.13)$$

For this state Theorem 5 poses the same bound given by Eq. (6.11). By calculations, we further obtain $\phi(\mathbf{M}|\varphi_3) = \langle \varphi_3, \mathbf{M}_3 \varphi_3 \rangle = 2/(\sqrt{2} + 1)$ and $\phi(\mathbf{N}|\varphi_3) = \langle \varphi_3, \mathbf{N}_1 \varphi_3 \rangle = \langle \varphi_3, \mathbf{N}_2 \varphi_3 \rangle = 1/2$. Corollary 8 then gives

$$H_\alpha(\mathbf{M}|\varphi_3) + H_\beta(\mathbf{N}|\varphi_3) \geq \ln(\sqrt{2} + 1). \quad (6.14)$$

The right-hand side of Eq. (6.11) is less than the right-hand side of Eq. (6.14). So, for state φ_3 the entropic relation (5.13) provides more stronger bound than the entropic relation (5.4). The difference between these bounds also is $\ln(\sqrt{2} + 1) - \ln 2 \approx 0.188$. To sum up, we see that both the entropic uncertainty relations (5.4) and (5.13) are independently significant.

Finally, we consider the state-independent bound for a single POVM. For projective measurement the trivial lower bound on the entropy is zero. This bound can exactly be reached. For POVM measurement an analogue is ensured by Eq. (5.17). Let \mathbf{M}_0 be a POVM element such that $\bar{\phi}(\mathbf{M}) = \|\mathbf{M}_0\|$. It is known that $H_\infty(p) = -\ln p_{\max}$ [47]. Thus, in the case $\alpha \gg 1$ the lower bound (5.17) is approximately reached for an eigenstate of \mathbf{M}_0 .

A Proof of Proposition 2

In this appendix we prove Eq. (3.5). Instead of $p_i^{(\psi)}$ and $q_j^{(\psi)}$, we shall further write p_i and q_j respectively. For those values of labels i and j that satisfy $\|\mathbf{P}_i\boldsymbol{\psi}\| \neq 0$ and $\|\mathbf{Q}_j\boldsymbol{\psi}\| \neq 0$ we define vectors

$$\mathbf{u}_i := \|\mathbf{P}_i\boldsymbol{\psi}\|^{-1} \mathbf{P}_i\boldsymbol{\psi} , \quad (\text{A.1})$$

$$\mathbf{v}_j := \|\mathbf{Q}_j\boldsymbol{\psi}\|^{-1} \mathbf{Q}_j\boldsymbol{\psi} . \quad (\text{A.2})$$

With no loss of generality, we can mean that $1 \leq i \leq m$ and $1 \leq j \leq n$. The \mathbf{u}_i 's and the \mathbf{v}_j 's form the two orthonormal sets. In general, these sets are not complete in the space \mathcal{H} . By definition, put

$$t_{ij} := \langle \mathbf{u}_i , \mathbf{v}_j \rangle . \quad (\text{A.3})$$

According to Eqs. (2.10) and (A.3),

$$y_i(x) = \langle \mathbf{u}_i , \mathbf{w} \rangle , \quad (\text{A.4})$$

where $\mathbf{w} := \sum_j x_j \mathbf{v}_j$ by definition. It is clear that the vector $\sum_i y_i \mathbf{u}_i$ is orthogonal projection of \mathbf{w} onto the subspace spanned by \mathbf{u}_i 's. So we get

$$\sum_i |y_i|^2 \leq \|\mathbf{w}\|^2 . \quad (\text{A.5})$$

On the other hand, due to the definition of \mathbf{w} and $\langle \mathbf{v}_j , \mathbf{v}_k \rangle = \delta_{jk}$ we have

$$\|\mathbf{w}\|^2 = \sum_j |x_j|^2 . \quad (\text{A.6})$$

Therefore, the condition (2.12) is satisfied for all $\mathbf{x} \in \mathbb{C}^n$. So we can apply Eq. (2.13). We shall now use this result for the values

$$y_i = \|\mathbf{P}_i\boldsymbol{\psi}\| , \quad (\text{A.7})$$

$$x_j = \|\mathbf{Q}_j\boldsymbol{\psi}\| . \quad (\text{A.8})$$

Both the PVM's $\{\mathbf{P}_i\}$ and $\{\mathbf{Q}_j\}$ satisfy the completeness relation. Substituting this in the identity $\boldsymbol{\psi} = \mathbf{I}\boldsymbol{\psi}$, in terms of the above values we have

$$\boldsymbol{\psi} = \sum_k y_k \mathbf{u}_k = \sum_j x_j \mathbf{v}_j . \quad (\text{A.9})$$

Combining the relation $\langle \mathbf{u}_i, \mathbf{u}_k \rangle = \delta_{ik}$ with Eq. (A.9), we get

$$y_i = \sum_j \langle \mathbf{u}_i, \mathbf{v}_j \rangle x_j . \quad (\text{A.10})$$

Thus, the values given by Eqs. (A.7) and (A.8) are really connected by Eq. (2.10) with the matrix elements (A.3). Further, $p_i = |y_i|^2$ and $q_j = |x_j|^2$ due to Eqs. (3.3) and (3.4). Let us put $a = 2\alpha$ and $b = 2\beta$. Squaring Eq. (2.13), after substitutions we obtain

$$S_\alpha(p) \leq \eta^{2(1-\beta)/\beta} S_\beta(q) , \quad (\text{A.11})$$

where $1/\alpha + 1/\beta = 2$, $1/2 < \beta < 1$ and

$$\eta = \max_{ij} |\langle \mathbf{u}_i, \mathbf{v}_j \rangle| . \quad (\text{A.12})$$

Using the definition of $H_\alpha(p)$ by Eq. (2.1), we get

$$\ln S_\alpha(p) = \frac{1-\alpha}{\alpha} H_\alpha(p) . \quad (\text{A.13})$$

In the same way the quantities $S_\beta(q)$ and $H_\beta(q)$ are related. It then follows from Eq. (A.11) and $(1-\beta)/\beta > 0$ that

$$\frac{(1-\alpha)\beta}{\alpha(1-\beta)} H_\alpha(p) \leq 2 \ln \eta + H_\beta(q) . \quad (\text{A.14})$$

When $1/\alpha + 1/\beta = 2$ and $\alpha, \beta \neq 1$, the multiplier of $H_\alpha(p)$ in Eq. (A.14) is equal to (-1) . So we can rewrite Eq. (A.14) as

$$H_\alpha(p) + H_\beta(q) \geq -2 \ln \eta . \quad (\text{A.15})$$

For projectors we clearly have $\mathbf{P}_i = \mathbf{P}_i^{1/2}$ and $\mathbf{Q}_j = \mathbf{Q}_j^{1/2}$. So due to Eqs. (A.1) and (A.2) the right-hand side of Eq. (A.12) is equal to $f(\mathbf{P}, \mathbf{Q}|\psi)$. This concludes the proof for $\alpha > \beta$. By permutation of the two PVM's, we recover the case when the order of entropy of PVM $\{\mathbf{Q}_j\}$ is greater than the order of entropy of PVM $\{\mathbf{P}_i\}$.

B Naimark's extension and related questions

We shall now describe the version of Naimark's extension stated in Ref. [40]. Let $\{\mathbf{E}_i\}$ be a set of positive semidefinite matrices satisfying

$$\sum_i \mathbf{E}_i = \mathbf{I}_{\mathcal{H}} , \quad (\text{B.1})$$

where $\mathbf{I}_{\mathcal{H}}$ is the identity operator in the space \mathcal{H} . Naimark proved that each generalized resolution of the identity can be realized as an orthogonal resolution of the identity for the enlarged space $\tilde{\mathcal{H}}$ which contains \mathcal{H} as a subspace [36, 37, 41]. Let us define

$$\tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{L} , \quad (\text{B.2})$$

where \mathcal{L} is a space of needed dimensionality. As it is interpreted by Parthasarathy [40], we can build partitioned matrices of the form

$$\tilde{\mathbf{E}}_i := \begin{bmatrix} \mathbf{E}_i & \mathbf{R}_i \\ \mathbf{R}_i^\dagger & \mathbf{L}_i \end{bmatrix} , \quad (\text{B.3})$$

so that the $\tilde{\mathbf{E}}_i$'s are orthogonal projections in the enlarged space $\tilde{\mathcal{H}}$ and

$$\sum_i \tilde{\mathbf{E}}_i = \tilde{\mathbf{I}} . \quad (\text{B.4})$$

Here the matrix $\tilde{\mathbf{I}}$ represents the identity operator in the space $\tilde{\mathcal{H}}$. In Eq. (B.3) the orders of submatrices \mathbf{R}_i and \mathbf{L}_i should be clear from the context. An arbitrary vector in the enlarged space is represented by the column

$$\tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \mathbf{z} \end{bmatrix} \quad (\text{B.5})$$

with $\mathbf{u} \in \mathcal{H}$ and $\mathbf{z} \in \mathcal{L}$. The entries of this column are components of $\tilde{\mathbf{u}}$ with respect to the orthonormal basis in $\tilde{\mathcal{H}}$ that is obtained by extension of the initial basis in \mathcal{H} .

To each $\boldsymbol{\psi} \in \mathcal{H}$ assign the vector $\tilde{\boldsymbol{\psi}} \in \tilde{\mathcal{H}}$ defined by

$$\tilde{\boldsymbol{\psi}} := \begin{bmatrix} \boldsymbol{\psi} \\ \mathbf{0} \end{bmatrix} . \quad (\text{B.6})$$

Here and below $\mathbf{0}$ denotes the matrix of needed order consisting of all zeros. Following the rules of block multiplication, we have

$$\tilde{\psi}^\dagger \tilde{\mathbf{E}}_i \tilde{\psi} = [\psi^\dagger \quad \mathbf{0}] \begin{bmatrix} \mathbf{E}_i \psi \\ \mathbf{R}_i^\dagger \psi \end{bmatrix} = \psi^\dagger \mathbf{E}_i \psi . \quad (\text{B.7})$$

In other words, the probability of getting outcome i , equal to

$$\langle \tilde{\psi} , \tilde{\mathbf{E}}_i \tilde{\psi} \rangle = \langle \psi , \mathbf{E}_i \psi \rangle , \quad (\text{B.8})$$

is not changed under the made extension.

Let $\{\mathbf{G}_j\}$ be another resolution of the identity for the space \mathcal{H} . To each \mathbf{G}_j assign the operator $\tilde{\mathbf{G}}_j$ acting on the space $\tilde{\mathcal{H}}$. In the matrix representation, we define these operators as follows:

$$\tilde{\mathbf{G}}_1 := \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\mathcal{L}} \end{bmatrix} , \quad \tilde{\mathbf{G}}_j := \begin{bmatrix} \mathbf{G}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (j \neq 1) . \quad (\text{B.9})$$

Here the identity matrix $\mathbf{I}_{\mathcal{L}}$ of corresponding order describes the action of the identity in the subspace \mathcal{L} . Because the \mathbf{G}_j 's form a resolution of the identity for the space \mathcal{H} , we then have

$$\sum_j \tilde{\mathbf{G}}_j = \tilde{\mathbf{I}} . \quad (\text{B.10})$$

Further, for all $\tilde{\mathbf{u}} \in \tilde{\mathcal{H}}$ there holds

$$\tilde{\mathbf{u}}^\dagger \tilde{\mathbf{G}}_j \tilde{\mathbf{u}} = [\mathbf{u}^\dagger \quad \mathbf{z}^\dagger] \begin{bmatrix} \mathbf{G}_j \mathbf{u} \\ \delta_{j1} \mathbf{z} \end{bmatrix} = \mathbf{u}^\dagger \mathbf{G}_j \mathbf{u} + \delta_{j1} \mathbf{z}^\dagger \mathbf{z} , \quad (\text{B.11})$$

So each $\tilde{\mathbf{G}}_j$ is positive semidefinite due to the positive semidefiniteness of the \mathbf{G}_j 's. Therefore, the set $\{\tilde{\mathbf{G}}_j\}$ is a resolution of the identity for the space $\tilde{\mathcal{H}}$. In addition, if the resolution $\{\mathbf{G}_j\}$ is orthogonal then the resolution $\{\tilde{\mathbf{G}}_j\}$ is also orthogonal. By Eq. (B.11), for any state of the form (B.6) we have

$$\langle \tilde{\psi} , \tilde{\mathbf{G}}_j \tilde{\psi} \rangle = \tilde{\psi}^\dagger \tilde{\mathbf{G}}_j \tilde{\psi} = \psi^\dagger \mathbf{G}_j \psi = \langle \psi , \mathbf{G}_j \psi \rangle . \quad (\text{B.12})$$

Thus, for second measurement the probability of getting outcome j is also not changed under the made extension. To sum up, we can say the following. Starting with the two POVM measurements $\{\mathbf{E}_i\}$ and $\{\mathbf{G}_j\}$, we have

constructed the two measurements $\{\tilde{E}_i\}$ and $\{\tilde{G}_j\}$ in the enlarged space $\tilde{\mathcal{H}}$. But the first measurement $\{\tilde{E}_i\}$ is now projective.

For any positive semidefinite operator there exists a unique positive square root. Further, each positive semidefinite operator is Hermitian. Using these facts and the definition of the norm, we get

$$||E_i^{1/2}\psi||^2 = \langle E_i^{1/2}\psi, E_i^{1/2}\psi \rangle = \langle \psi, E_i\psi \rangle . \quad (B.13)$$

Combining this with Eq. (B.8) finally gives

$$||\tilde{E}_i^{1/2}\tilde{\psi}|| = ||E_i^{1/2}\psi|| \quad (B.14)$$

for every state of the form (B.6). In the same manner, due to Eq. (B.12) we obtain

$$||\tilde{G}_j^{1/2}\tilde{\psi}|| = ||G_j^{1/2}\psi|| . \quad (B.15)$$

By matrix calculations, we also have

$$\tilde{\psi}^\dagger \tilde{E}_i \tilde{G}_j \tilde{\psi} = [\psi^\dagger \quad 0] \begin{bmatrix} E_i & R_i \\ R_i^\dagger & L_i \end{bmatrix} \begin{bmatrix} G_j \psi \\ 0 \end{bmatrix} = \psi^\dagger E_i G_j \psi . \quad (B.16)$$

By Hermiticity of the POVM elements, in terms of inner products one gives

$$\langle \tilde{E}_i \tilde{\psi}, \tilde{G}_j \tilde{\psi} \rangle = \langle E_i \psi, G_j \psi \rangle . \quad (B.17)$$

Together with Eqs. (B.14) and (B.15) the last equality implies

$$f(\tilde{E}, \tilde{G} | \tilde{\psi}) = f(E, G | \psi) . \quad (B.18)$$

Equation (B.18) is valid for arbitrary state vector of the form (B.6). Here the one fact should be pointed out. By calculations,

$$\tilde{\psi}^\dagger \tilde{E}_i \tilde{E}_i \tilde{\psi} = \psi^\dagger E_i E_i \psi + \psi^\dagger R_i R_i^\dagger \psi . \quad (B.19)$$

The last equality implies that $||\tilde{E}_i \tilde{\psi}|| \neq ||E_i \psi||$ in general. It is for this reason that the square roots of operators are inserted into the fraction denominator in the right-hand side of Eq. (2.6).

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